On the Distance Function between Two Confocal
Keplerian Orbits

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Abstract

We present analytical and numerical explorations of the problem of the minimum distance
between two elliptical orbits with common focus. We can state the general solution
in the form of a polynomial of degree eight in \( \{ \cos E_1, \cos E_2 \} \), where \( E \) is the eccentric
anomaly. By use of the eccentric anomaly as the independent variable, the algebraic
complexity of the derivatives required for location of the stationary points is reduced.
Numerically, one can immediately use the Kepler equation in combination with a fast
and simple search algorithm — which steps in units of orbital periods — to quickly
determine the times of close approach between two objects. Thus, one application of
these results is as a relatively fast filter for selecting (for closer scrutiny later) asteroids
that will suffer a close approach to a given planet or other asteroid within a specified
time segment, without the need for numerical integrations of the differential equations
of motion.

Subject headings: celestial mechanics—methods: analytical—methods: numerical—minor
planets, asteroids—planets and satellites: general

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1 Introduction

The minimum distance between two Keplerian orbits with common focus pertains to many problems in astronomy and celestial mechanics, belonging to such broad areas as meteorites (intersections of comet debris with planetary orbits), NEOs (close approaches of asteroids with Earth, as well as other planets), asteroid masses (using close approaches to determine masses), and Earth-orbiting space debris (collisions between junk and crewed vehicles or valuable hardware).

For convenience, we define a function $F(x_1, x_2)$ of the distance between two particles on confocal Keplerian orbits as the square of the distance,

$$ F(x_1, x_2) = |\vec{r}_2(x_2) - \vec{r}_1(x_1)|^2 $$

(1)

where $\vec{r}_k$ are position vectors from the orbit focus and $x_k$ are independent (angular) variables locating the position of the respective particles along their orbital paths at a given time. We may choose the independent variables to be the true anomalies, $x_k = \theta_k$, or the eccentric anomalies, $x_k = E_k$, but we are not limited necessarily to these choices. The distance function itself is easily written down in terms of the orbital elements of each body, but the minimization of $F(x_1, x_2)$ is analytically intractible.

Various authors have discussed the minimization problem. Sitarski (1968)[7] [...].

Bonanno (2000)[2] [...]

Kholshevnikov and Vassiliev (1999)[6] [...]

Gronchi (2002)[5] [...]

2 Stationary Points of the Distance Function

2.1 Formal Statement of the Problem

2.2 Location of the Distance Minima by Numerical Means

3 A Fast Numerical Approach for Finding Approximate Times of Close Approach

3.1 A Simple Yet Useful Algorithm

3.2 Example: Asteroid-Asteroid Close Approaches

4 Conclusions
1.1 The Distance Function in Orbital Elements

The reduced-mass solution of the ideal gravitational two-body problem is a Keplerian orbit with semimajor axis $a$ and eccentricity $e$ that can be written either as

$$\vec{r'}(\theta) = \frac{a(1-e^2)}{1 + ec\cos\theta} Q(\Omega, \iota, \omega) \cdot \vec{q'}(\theta)$$

or

$$\vec{r}(E) = aQ(\Omega, \iota, \omega) \cdot \vec{q}(E)$$

where the vectors

$$\vec{q'}(\theta) = \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix}$$

$$\vec{q}(E) = \begin{bmatrix} \cos E - e \\ \sqrt{1 - e^2\sin E} \\ 0 \end{bmatrix}$$

specify the position along the orbit curve, and the orbit orientation matrix is

$$Q(\Omega, \iota, \omega) = \begin{bmatrix} \cos \Omega \cos \omega - \sin \Omega \sin \omega \cos \iota & -\cos \Omega \sin \omega - \sin \Omega \cos \omega \cos \iota & \sin \Omega \sin \iota \\ \sin \Omega \cos \omega + \cos \Omega \sin \omega \cos \iota & -\sin \Omega \sin \omega + \cos \Omega \cos \omega \cos \iota & -\cos \Omega \sin \iota \\ \sin \omega \sin \iota & \cos \omega \sin \iota & \cos \iota \end{bmatrix}$$

This formulation cleanly separates the orbit scale $a$, the shape of the orbit $e$ plus the position of the particle at a given time $\vec{q'}$, and the orientation of the orbit in space $Q(\Omega, \iota, \omega)$. The eccentric and true anomalies are functions of time (1) through Kepler’s equation

$$E - es\sin E = n(t - \tau)$$

where $n_k$ is the mean motion given by the two-body relation $n_k^2 a_k^3 = G(m_c + m_k)$ and $\tau$ is the time of pericenter passage ($m_c$ is the central body mass), and (2) by the geometric relations

$$\cos E = \frac{e + \cos\theta}{1 + ec\cos\theta} \quad \sin E = \frac{\sqrt{1 - e^2\sin\theta}}{1 + ec\cos\theta}$$
Therefore, in terms of the true anomalies, (1) can be written

\[ F(\theta_1, \theta_2) = r_1^2(\theta_1) + r_2^2(\theta_2) + 2 \{ \cos \Omega_2 \cos (\theta_1 + \omega_1) \sin (\theta_2 + \omega_2) - \cos \Omega_1 \sin (\theta_1 + \omega_1) \cos (\theta_2 + \omega_2) \} \sin \Delta \Omega \\
\]

where \( \Delta \Omega = \Omega_2 - \Omega_1 \). (See Figure 1 for definitions of the orbital elements.) Equivalently, we may expand the trig functions and write

\[ F(\theta_1, \theta_2) = r_1^2(\theta_1) + r_2^2(\theta_2) - 2 \{ (A_1 \sin \theta_2 + A_2 \cos \theta_2) \sin \theta_1 + (A_3 \sin \theta_2 + A_4 \cos \theta_2) \cos \theta_1 \} r_1(\theta_1) r_2(\theta_2) \]

where

\[
\begin{align*}
A_1 &= (\cos \zeta \cos \omega_2 - \sin \omega_1 \cos \beta) \\
A_2 &= ( \cos \zeta \sin \omega_2 + \sin \omega_1 \cos \gamma) \\
A_3 &= (\cos \zeta \cos \omega_2 - \sin \omega_1 \cos \beta) \\
A_4 &= (\cos \zeta \sin \omega_2 + \sin \omega_1 \cos \gamma) \\
cos \zeta &= \sin \Delta \Omega \cos \omega_2 + \sin \omega_2 \cos \Delta \Omega \\
\sin \beta &= \sin \Delta \Omega \cos \omega_2 + \sin \omega_2 \cos \Delta \Omega \\
\sin \gamma &= -\cos \omega_2 \cos \Delta \Omega + \sin \omega_2 \sin \Delta \Omega \\
\end{align*}
\]

The forms of (9) or (10) are not particularly conducive for our goal of determination of the stationary points of the distance function (Section 2). On the other hand, in terms of the eccentric anomalies, (1) can be written

\[ F(E_1, E_2) = (1 - e_1 \cos E_1)^2 a_1^2 + (1 - e_2 \cos E_2)^2 a_2^2 - 2 \{ \sqrt{1 - e_2^2} \sin E_2 [B_1 \sqrt{1 - e_1^2} \sin E_1 + B_2 (\cos E_1 - e_1)] + (\cos E_2 - e_2) [B_3 \sqrt{1 - e_1^2} \sin E_1 + B_4 (\cos E_1 - e_1)] \} a_1 a_2 \]

where

\[
B_k = c_k \cos \Delta \Omega + s_k \sin \Delta \Omega + d_k, \quad k = 1..4
\]

\[
\begin{align*}
c_1 &= \cos \omega_1 \cos \omega_2 + \sin \omega_1 \sin \omega_2 \\
c_2 &= \cos \omega_1 \cos \omega_2 - \sin \omega_1 \sin \omega_2 \\
c_3 &= \cos \omega_1 \cos \omega_2 - \sin \omega_1 \cos \omega_2 \\
c_4 &= \cos \omega_1 \cos \omega_2 + \sin \omega_1 \cos \omega_2
\end{align*}
\]
\[
\begin{aligned}
    s_1 &= \cos \omega_2 \cos \omega_1 - \cos \omega_1 \sin \omega_2 \\
    s_2 &= -\cos \omega_1 \sin \omega_2 - \cos \omega_2 \cos \omega_1 \\
    s_3 &= \cos \omega_1 \cos \omega_2 + \cos \omega_1 \sin \omega_2 \\
    s_4 &= \cos \omega_1 \cos \omega_2 - \cos \omega_2 \cos \omega_1 \\
\end{aligned}
\] (15)

\[
\begin{aligned}
    d_1 &= \sin \omega_1 \cos \omega_1 \cos \omega_2 \\
    d_2 &= \sin \omega_1 \sin \omega_2 \cos \omega_1 \cos \omega_2 \\
    d_3 &= \sin \omega_1 \sin \omega_2 \cos \omega_1 \sin \omega_2 \\
    d_4 &= \sin \omega_1 \sin \omega_2 \sin \omega_1 \sin \omega_2 \\
\end{aligned}
\] (16)

At first glance, it seems that by (12) we have increased the complexity of the expression for the distance function and done ourselves no favors. However, the functions for the radial distances, \( r_k(\theta) \), have disappeared along with their unpleasant denominators. More importantly, using the eccentric anomalies thus leads to less-complicated derivatives of the distance function due to the removal of the denominator terms \( 1 + e \cos \theta \).

### 1.2 Reduction to Relative Elements

![Figure 1: Relative orbital elements.](image)

We may consider the problem with respect to a coordinate system \([X', Y', Z']\) whose \(X'Y'\) plane is coincident with the orbital plane of \(m_1\) and whose \(X'\) axis is in the direction of the pericenter of \(m_1\) (i.e., the Laplace-Runge-Lenz vector \( \vec{v} \times \vec{h} - \mu \frac{\vec{r}}{r^2} \), where \( \vec{h} = \vec{r} \times \vec{v} \) and \( \mu = G(m_c + m) \)). The orientation of the orbit of \(m_2\) with respect to this coordinate system is then specified by the relative orbit orientation elements, \(\{\Omega, \omega, \iota\}\), as shown in Figure 1.
We then have the following relations from the spherical triangle outlined in Figure 1:

\[
\begin{align*}
\cos \Delta \Omega &= \cos(\omega_1 + \Omega) \cos(\omega_2 - \omega) + \sin(\omega_1 + \Omega) \sin(\omega_2 - \omega) \cos \iota \\
\cos(\omega_2 - \omega) &= \cos \Delta \Omega \cos(\omega_1 + \Omega) + \sin \Delta \Omega \sin(\omega_1 + \Omega) \cos \iota_1 \\
\cos(\omega_1 + \Omega) &= \cos \Delta \Omega \cos(\omega_2 - \omega) - \sin \Delta \Omega \sin(\omega_2 - \omega) \cos \iota_2 \\
\sin(\omega_1 + \Omega) \sin \iota_1 &= \sin(\omega_2 - \omega) \sin \iota_2 \\
\sin(\omega_1 + \Omega) \sin = &\sin \Delta \Omega \sin \iota_2 \\
\sin \Delta \Omega \sin \iota_1 &= \sin(\omega_2 - \omega) \sin \iota \\
\end{align*}
\]

where the last three equations constitute two independent relations. Hence, we may eliminate five parameters. As a practical matter, one may therefore calculate the relative elements from two sets of known elements by extracting from eqs. (17) the following:

\[
\begin{align*}
\cos \Omega \sin \iota &= \sin \omega_1 \sin \iota_2 \sin \Delta \Omega - \cos \omega_1 (\sin \iota_1 \cos \iota_2 - \cos \iota_1 \sin \iota_2 \cos \Delta \Omega) \\
\sin \Omega \sin \iota &= \cos \omega_1 \sin \iota_2 \sin \Delta \Omega + \sin \omega_1 (\sin \iota_1 \cos \iota_2 - \cos \iota_1 \sin \iota_2 \cos \Delta \Omega) \\
\cos \omega \sin \iota &= \sin \iota_1 \sin \Delta \Omega + \cos \omega_2 (\cos \iota_1 \sin \iota_2 - \sin \iota_1 \cos \iota_2 \cos \Delta \Omega) \\
\sin \omega \sin \iota &= -\cos \omega_2 \sin \iota_1 \sin \Delta \Omega + \sin \omega_2 (\cos \iota_1 \sin \iota_2 - \sin \iota_1 \cos \iota_2 \cos \Delta \Omega)
\end{align*}
\]

(18)

The relative inclination is then found from the remainder equation, which after some simplification can be written

\[
(1 - \cos^2 \iota_2) (\cos \iota + \cos \iota_1 \cos \iota_2) (\cos \iota_1 \cos \iota_2 + \sin \iota_1 \sin \iota_2 \cos \Delta \Omega - \cos \iota) = 0
\]

(19)

This is of fourth order in \( \cos \iota \), but as we see it factors very conveniently. The solutions \( \cos \iota = \pm 1 \) and \( \cos \iota = -\cos \iota_1 \cos \iota_2 \) are special cases requiring \( \iota = b\pi \), where \( b \in \{-1, 0, 1\} \).

The generally useful solution is therefore

\[
\cos \iota = \cos \iota_1 \cos \iota_2 + \sin \iota_1 \sin \iota_2 \cos \Delta \Omega
\]

(20)

which, with eqs. (18), allow us to convert to relative elements. One notices from Figure 1 that eqs. (18) eliminate the two sides of the spherical triangle involving unknowns, while (20) removes the interior angle between them. In fact, we recognize (20) as the interior angle analog of the more familiar law of cosines for spherical triangles (e.g., [9]). Finally, we may define

\[
\alpha \equiv \frac{a_2}{a_1}
\]

(21)

Then the free parameters of the problem have been reduced to the set \( \{\alpha, e_1, e_2, \iota, \omega, \Omega; E_1, E_2\} \), with \( E_1 \) and \( E_2 \) the independent variables specifying location on the orbit curves and the others setting the geometry (shapes and relative orientation).
In terms of relative elements, (I) becomes

\[ F(E_1, E_2) = a_1^2 | \alpha Q(\Omega, \iota, \omega) \cdot \vec{q}(E_2) - \vec{q}(E_1) |^2 \]  

which can be written in the dimensionless form

\[ \frac{1}{a_{12}} F(E_1, E_2) = (1 - e_1 \cos E_1)^2 + \alpha^2 (1 - e_2 \cos E_2)^2 \]

\[ -2\alpha \left\{ \sqrt{1 - e_2^2 \sin E_2} \left[ C_1 \sqrt{1 - e_1^2 \sin E_1} + C_2 (\cos E_1 - e_1) \right] \right. \]

\[ + (\cos E_2 - e_2) \left\{ C_3 \sqrt{1 - e_1^2 \sin E_1} + C_4 (\cos E_1 - e_1) \right\} \]  

where

\[ C_1 = \sin \Omega \sin \omega - \cos \Omega \cos \omega \cos \iota \]

\[ C_2 = \cos \Omega \sin \omega + \sin \Omega \cos \omega \cos \iota \]

\[ C_3 = -\sin \Omega \cos \omega - \cos \Omega \sin \omega \cos \iota \]

\[ C_4 = -\cos \Omega \cos \omega + \sin \Omega \sin \omega \cos \iota \]  

Eq. (23) is the same form as (12) but with considerably less-complicated coefficients.

2 Stationary Points of the Distance Function

2.1 Formal Statement of the Problem

The distance function defines a family of two-dimensional differentiable manifolds. Minima of the distance function will correspond to stationary points at the “valleys” of the surfaces described by \( F(E_1, E_2) = 0 \). The independent variables we find convenient are \( \{ E_1, E_2 \} \), while the manifolds are parameterized by \( \{ \alpha, e_1, e_2, \iota, \omega, \Omega \} \). The extrema of a particular surface may be found by setting the derivatives of the distance function equal to zero (i.e., finding the locations where the slope of the tangent plane is zero). There are in general two minima and two maxima, so the surfaces commonly have two peaks and two valleys over the domain \( \{ E_1, E_2 \} \in [0, 2\pi] \). Taking the derivatives of (23), we have

\[ \frac{1}{2a_{12}} \frac{\partial F(E_1, E_2)}{\partial E_1} = \frac{1}{\alpha} (1 - e_1 \cos E_1) e_1 \sin E_1 \]

\[ - \left[ \sqrt{1 - e_2^2 \sin E_2} (C_1 \sqrt{1 - e_1^2 \sin E_1} - C_2 \sin E_1) \right] \]

\[ \frac{1}{2a_{12}} \frac{\partial F(E_1, E_2)}{\partial E_2} = \alpha (1 - e_2 \cos E_2) e_2 \sin E_2 \]

\[ - \left\{ \sqrt{1 - e_2^2 \cos E_2} \left[ C_1 \sqrt{1 - e_1^2 \sin E_1} + C_2 (\cos E_1 - e_1) \right] \right. \]

\[ - \sin E_2 \left\{ C_3 \sqrt{1 - e_1^2 \sin E_1} + C_4 (\cos E_1 - e_1) \right\} \]  

We wish to locate, for a particular set of parameters \( \vec{p} = \{ \alpha, e_1, e_2, \iota, \omega, \Omega \} \), the values of \( \{ E_1, E_2 \} \) that minimize the distance function. We can then use Kepler’s equation to find...
the corresponding times of closest approach. Equations (25) appear to be difficult, and
indeed they are. Formally, we can set \( \sin E_k = \pm \sqrt{1 - \cos^2 E_k} \) and write the equations as
polynomials in \( \{ \cos E_1, \cos E_2 \} \). The result is a pair of bivariate polynomials of order eight,
of the form

\[
\sum_{0 \leq k + m \leq 8} D_{k,m}^{(i)}(\vec{p}) \cos^m E_1 \cos^k E_2 = 0, \quad i \in \{1, 2\}
\]

where the coefficients \( D_{k,m}^{(i)}(\vec{p}) \) are rather bulky and unenlightening. The order is due
to the need to square each equation twice to remove the square roots, which, in concert
with \( \cos E_k \sin E_j \) terms originally outside the square roots, results in terms of order eight.
Thus, the problem is analytically intractable except for special cases. There appears to be no
simpler reduction of the general problem in this coordinate geometry. Gronchi [5] performs a
change of variables to obtain a pair of bivariate polynomials of degree six. His transformation
is essentially a mapping from a circle to a line (equivalent to the stereographic coordinates
familiar to geographic mapmakers) and thus introduces singularities. For our purposes, that
approach, although clever and interesting, unfortunately does not lead to any significant
advantages here.

2.2 Location of the Distance Minima by Numerical Means

- 2D surface (pretty plot of eq. (23))
- review method of Sitarski and show why it is so outdated and inefficient
- Newton-Raphson
  - fast approximation for N-R starting points
  - \( x \) times faster than Sitarski

3 A Fast Numerical Approach for Finding Approximate Times
of Close Approach

3.1 A Simple Yet Useful Algorithm

We may, for particular parameter values (corresponding to real objects as transformed by
eqs. (18) and (20)), solve by numerical means the derivative equations (25) for specific
values of\( \{ E_1, E_2 \} \), viz.,

\[
\{ E_1(\vec{p}), E_2(\vec{p}) \left| \nabla_{E_1,E_2} F(E_1, E_2) = 0 \right. \}
\]

(27)
Of note is the fact that, once \( \{E_1, E_2\} \) corresponding to distance minima have been found, one may use Kepler’s equation (7) to quickly locate those times when both bodies are simultaneously at or very near their respective mean anomalies corresponding to the distance minima.

Write Kepler’s equation in the form

\[
E - e \sin E = M_0 + n (t - t_0)
\]

where \( M_0 \) is the mean anomaly of epoch \( t_0 \), and \( n \) is the mean motion. (Thus, from (7), \( M_0 = n(t_0 - \tau) \).) From numerically-determined solutions for the distance function stationary points, we have two values of \( E \) corresponding to the distance minima and, therefore, corresponding values of the mean anomaly and time, say \( \widetilde{M} \) and \( \widetilde{t} \). Define \( \Delta M = \widetilde{M} - M_0 \). Then the time at which a body is at a minimum-distance point is

\[
\widetilde{t} = t_0 + \frac{\Delta M}{n}
\]

If \( P \) is the orbital period, then successive times of passage through the minimum-distance point for each orbit separately are

\[
\tilde{t}_k = \tilde{t} + k \cdot P = t_0 + \frac{1}{n}(\Delta M + 2\pi k)
\]

with \( k \in \mathbb{Z} \). For a close approach to occur between two bodies, they must both be at or near their respective minimum-distance points within some small time interval, say \( \delta t \ll P \), of each other. Thus, we require

\[
|\tilde{t}_{k_2} - \tilde{t}_{k_1}| = \left| (t_{0_2} - t_{0_1}) + \frac{1}{n_2}(\Delta M_2 + 2\pi k_2) - \frac{1}{n_1}(\Delta M_1 + 2\pi k_1) \right| \leq \delta t
\]

or

\[
|\eta_1 \Delta t_0 + (\Delta M_2 - \eta \Delta M_1) - 2\pi(k_2 - \eta k_1)| \leq \eta_1 \delta t
\]

where

\[
\eta = \frac{n_2}{n_1} = \sqrt{\frac{m_c + m_2}{m_c + m_1}} \alpha^{-3/2} = \alpha^{-3/2} \left[ 1 + \frac{\epsilon_{2} - \epsilon_1}{2} - \frac{(\epsilon_{2} + 3\epsilon_1)(\epsilon_{2} - \epsilon_1)}{8} + \ldots \right]
\]

\( \epsilon_k = m_k/m_c, \) and \( \Delta t_0 = t_{0_2} - t_{0_1} \). If the epoch of initial coordinates of each body is the same, then \( \Delta t_0 = 0 \). Eq. (32) can be solved numerically in a simple-minded manner very quickly by just running through the integers \( \{k_1, k_2\} \) within some time span \( T \), with the \( k_i \).
constrained by \( \{ k_i \in \mathbb{Z} : k_i P_i \leq T, i = 1, 2 \} \), and listing the pairs, if any, such that (32) is satisfied.

We have reduced the problem of finding close approaches between two specific bodies to (1) the calculation of eccentric anomalies corresponding to the locations of orbit distance minima, followed by (2) a simple search for pairs of integers \( \{ k_1, k_2 \} \), within some range of time, for which inequality (32) is satisfied. Tests within the Maple symbolic algebra environment (not known for its numerical speed or efficiency) shows that this method is relatively fast — orders of magnitude faster than integrating the N-body differential equations of motion and searching/monitoring for close approaches.

### 3.2 Example: Asteroid-Asteroid Close Approaches

The table below shows a small sample of asteroid-asteroid encounters calculated using this method. The first 100 numbered asteroids were taken as a proof-of-concept data set, and all \( N(N - 1)/2 = 4950 \) possible pairs were searched over a 60-year span, with no attempt at optimization. The three columns in the table are the pair identification, the time of close encounter in years from \([\text{DATE}]\), and the minimum possible close approach distance in AU. Thus, this appears to be a viable method and worth further development and optimizations.

### 4 Conclusions

### References


